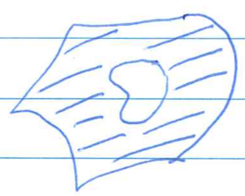
Lecture 15

MATH 2020A
Fall, 2020

- curves, vector fields
- line integral of the second kind

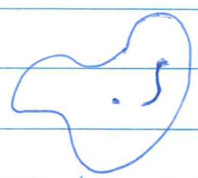
First let's review some definitions.

A region is always the set of points bounded by one or several piecewise C^1-curves in $\mathbb{R}^2$. A region is the set of points bdd by one or several piecewise C^1-surfaces in $\mathbb{R}^3$. It is possible to remove some points in a 2-D or 3-D region.



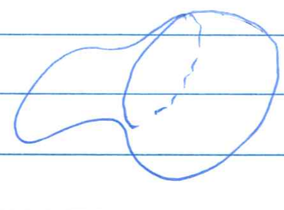
bdd by 2 curves

$n=2$

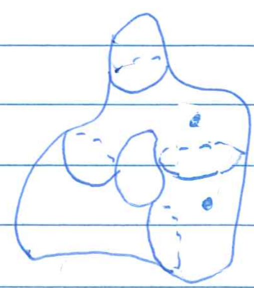


bdd by one curve
one pt and one curve removed.

$n=2$



$n=3$



2 points are removed

$n=3$

A region consists of interior pts and boundary pts. It is always bdd.

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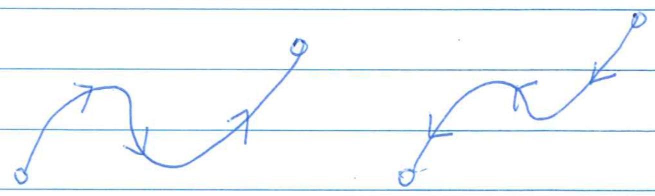
A parametric curve is a continuous map : $[a,b] \to \mathbb{R}^2, \mathbb{R}^3$. A regular parametric curve $\gamma : [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ satisfies $\gamma'(t)$ contin., $\forall t$ and $|\gamma'(t)| > 0$. A piecewise regular parametric curve $\gamma : [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ is a parametric curve so that its restriction on $[a, t_1], [t_1, t_2], \ldots, [t_{n-1}, b]$ are regular.

A (geometric) curve C is a subset in $\mathbb{R}^2, \mathbb{R}^3$ such that $\exists$ a parametric curve $\gamma : [a,b] \to C$ which is regular, and bijective $[a,b)$. Such parametric curve is called a parmetrization of C. Any parametrization of C carries an orientation on C. When $\gamma : [a,b] \to C$, the new prametrization $\eta : [a,b] \to C$ given by $\eta(t) = \gamma(a+b-t)$.

$\eta(a)=\gamma(b)$, $\eta(b)=\gamma(a)$, reverse the orientation. One can show that there are exactly two possible orientations on a curve C. More precisely, let γ_1 and γ_2 be two parametrizations of the same C on $[a,b]$ and $[c,d]$ respectively. Then $\exists\, \varphi:[a,b]\to[c,d]$ C^1-bijection such that $\gamma_2(\varphi(t))=\gamma_1(t)$, $t\in[a,b]$, or $\gamma_2(\varphi(t))=\gamma_1(a+b-t)$.



Only 2 orientations on the same C.

When C represents an oriented curve (a curve with an orientat-) we denote its reverse oriented curve by $-C$.

The tangent of a parametric curve γ is given by $\gamma'(t)$. Its magnitude and direction are given respectively by

$$|\gamma'(t)| = \left(x'^2(t)+y'^2(t)\right)^{1/2} \qquad n=2,$$
$$= \left(x'^2(t)+y'^2(t)+z'^2(t)\right)^{1/2} \qquad n=3,$$

and

$$\frac{\gamma'(t)}{|\gamma'(t)|}.$$

When C is an oriented curve parametrized by γ, its unit tangent at $\gamma(t)$ is $\dfrac{\gamma'(t)}{|\gamma'(t)|}$.

We use notation

$$\vec{T}(\gamma(t)) \quad \text{or} \quad \hat{t}(\gamma(t))$$

to denote it. Clearly, it is indept of the parametrization, ie

$$\gamma_1:[a,b]\to\mathbb{R}^2,\mathbb{R}^3$$
$$\gamma_2:[c,d]\to\mathbb{R}^2,\mathbb{R}^3$$

two parametrizations with the same orientation.

Then $\gamma_2(\varphi(t))=\gamma_1(t)$ for $\varphi:[a,b]\to[c,d]$ C^1-bijection, since $\varphi(a)=c$, $\varphi(b)=d$, $\varphi'(t)>0$. thus

$$\gamma_1'(t)=\gamma_2'(z)\,\varphi'(t)\quad,\quad z=\varphi(t)$$

$$\therefore\ \frac{\gamma_1'(t)}{|\gamma_1'(t)|}=\frac{\gamma_2'(z)\,\varphi'(t)}{|\gamma_1'(z)|},$$

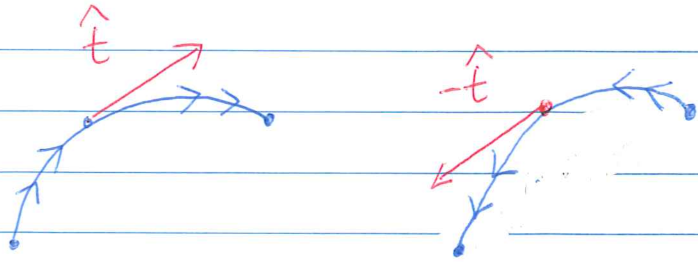
$$=\frac{\gamma_2'(z)\,\varphi'(t)}{|\gamma_2'(z)|\,|\varphi'(t)|}\qquad(\varphi'(t)=|\varphi'(t)|\ \because\ \varphi'>0)$$

$$=\frac{\gamma_2'(z)}{|\gamma_2'(z)|}.$$

On the other hand, the unit tangent of $-C$ is the negative of the unit tangent of C at the same point. Let $p=\gamma_2(t)=\gamma_1(a+b-t)$ be the same pt on C.

$$\gamma_2'(t)=-\gamma_1'(a+b-t)$$

$$\therefore\ \underbrace{\frac{\gamma_2'(t)}{|\gamma_2'(t)|}}_{\text{the unit tg of } -C \text{ at } \gamma_2(t)}=\underbrace{\frac{-\gamma_1'(a+b-t)}{|\gamma_1'(a+b-t)|}}_{\text{the unit tg of } C \text{ at } \gamma_1(a+b-t)}.$$



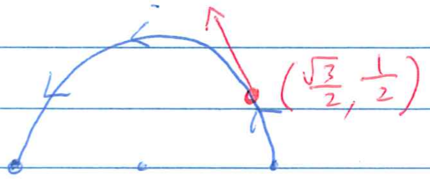
The pt on curve is $\gamma_2(t)=\gamma_1(a+b-t)$

e.g 1 Consider the following two parametrizations for the oriented semicircle (anti-clockwise).

$$\gamma_1 : [0,\pi] \to (\cos t, \sin t)$$

$$\gamma_2 : [-1,1] \to (-t, \sqrt{1-t^2})$$

Let's find out its unit tg at the pt $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ ie $\gamma_1\left(\frac{\pi}{6}\right)$ or $\gamma_2\left(-\frac{\sqrt{3}}{2}\right)$



Use γ_1 : $\gamma_1'(t) = (-\sin t, \cos t)$, $|\gamma_1'(t)| = \left((-\sin t)^2 + \cos^2 t\right)^{1/2} = 1$,

$$\therefore \frac{\gamma_1'(t)}{|\gamma_1'(t)|} = \frac{(-\sin t, \cos t)}{1} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ at } t = \frac{\pi}{6}$$

Use γ_2 : $\gamma_2'(t) = \left(-1, \frac{-t}{\sqrt{1-t^2}}\right)$, $|\gamma_2'(t)| = \left(1 + \frac{t^2}{1-t^2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{1-t^2}}$.

$$\frac{\gamma_2'(t)}{|\gamma_2(t)|} = \sqrt{1-t^2}\left(-1, \frac{-t}{\sqrt{1-t^2}}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ at } t = -\frac{\sqrt{3}}{2}.$$

$\therefore$ $\hat{t}$, or $\vec{T}$ are the same at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ on C.

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~~Vector fields on a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ are~~

$$\cancel{\vec{F}(x,y) = (P(x,y), Q(x,y))} \text{ ~~or~~ }$$

$$\cancel{= (P(x,y,z), Q(x,y,z), R(x,y,z))}$$

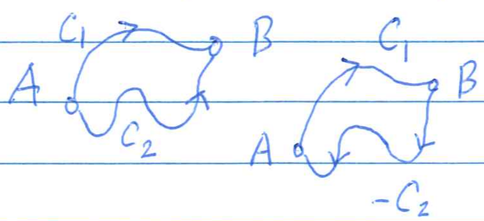
Let C_1 be a curve with parametrization $\gamma : [a,b]$ and and C_2 another curve with parametrization $\eta : [c,d]$. When $\gamma(b) = \eta(c)$, one can put C_1 and C_2 together to form a new curve C. It is continuous at $\gamma(b)$ ($\because \gamma(b) = \eta(c)$) but may not be differentiable there (unless $\gamma'(b) = \eta'(c)$). It is only piecewise regular even when γ and η are regular. We can also form a new parametrization of C, $\tilde{\gamma}$, on a single interval, but this is not always necessary.

Usually we denote C by $C_1 + C_2$. Similarly, one define

$$C_1 + C_2 + \cdots + C_n.$$

We also define $C_1 - C_2 = C_1 + (-C_2)$.

Let C_i, $i = 1, 2$, be two curves from A to B ($A, B \in \mathbb{R}^2, \mathbb{R}^3$), then $C_1 - C_2$ forms a closed curve.



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A vector field in a region (or in the interior of a region)

$$\vec{F}(x, y) = (P(x,y), Q(x,y)) \qquad n = 2$$

$$\vec{F}(x, y, z) = (P(x,y,z), Q(x,y,z), R(x,y,z)), \qquad n = 3.$$

$\vec{F}$ is conti/C^1 whenever its components are conti/C^1.

Some examples of vector fields were given in the last Lectures.

Here let's single out a class of special, but important vector fields.

A vector field $\vec{F}$ is called a gradient vector field if

$\exists$ function Φ in D such that

$$\vec{F} = \nabla \Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right) \text{ or}$$

$$= \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right).$$

Φ is called the potential of $\vec{F}$. It is unique up to the addition of an arbitrary constant.

A vector field in general depends on 2 unknows P, Q $(n=2)$ and 3 unknowns P, Q, R $(n=3)$. But a gradient v.f. only depends on a single unknown Φ, so it is very special. On the other hand, it is important because some basic v.f. in physics are gradient v.f.

e.g. 2 Any constant v.f. is gradient.

Let $\vec{F}(x,y,z) = (a,b,c)$ be constant v.f. then

$\Phi(x,y,z) = ax + by + cz$ is its potential.

e.g. 3 The gravitational v.f.

$$\vec{F} = -\frac{mMG}{r^3}\vec{r}, \quad \vec{r} = (x,y,z)$$

is a gradient v.f. For, let

$$\Phi(x,y,z) = \frac{mMG}{|r|} = \frac{mMG}{(x^2+y^2+z^2)^{1/2}}.$$

Then, by a direct check,

$$\frac{\partial \Phi}{\partial x}(x,y,z) = -\frac{mMG}{r^3}x,$$

$$\frac{\partial \Phi}{\partial y}(x,y,z) = -\frac{mMG}{r^3}y,$$

$$\frac{\partial \Phi}{\partial z}(x,y,z) = -\frac{mMG}{r^3}z, \text{ so}$$

$$\nabla\Phi = \vec{F}.$$

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Now, we come to define the line integral of a v.f. $\vec{F}$ along an oriented curve C.

First, recall that for a function defined along C, its line integral is

$$\int_C f\,ds = \lim_{\|P\|\to 0} \sum_j f(\gamma(t_j^*))\,|\gamma(t_j) - \gamma(t_{j-1})|,$$

where γ is a parametrization of C. This integral exists when f is continuous along C. Moreover, it is independent of the orientation induced by γ.

Now, given an oriented curve C and a v.f. along C, the line integral of the vector field $\vec{F}$ along C is defined to be

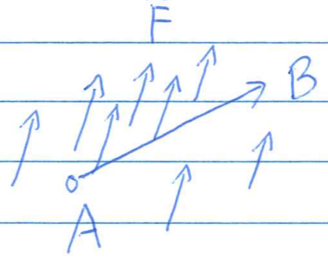
$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds \qquad (1)$$

where $\vec{T} = \gamma'(t)/|\gamma'(t)|$ the unit tangent of a parametrization γ which fits in the orientation of C. Note that $\vec{T}$ is independent of the choice of γ. However, when the orientation of C is reversed, $\vec{T}$ goes over to $-\vec{T}$, so we have

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}.$$

When $C = C_1 + \cdots + C_n$, C_j C^1-curve, define $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \cdots + \int_{C_n} \vec{F}\cdot d\vec{r}$.

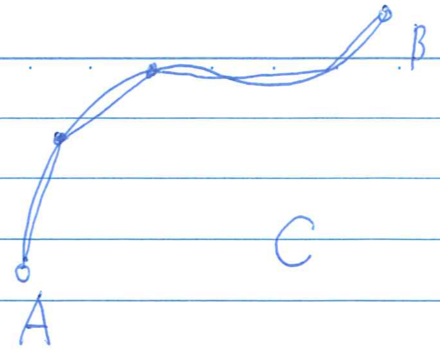
The motivation in the definition in (1) arises from physics. In mechanics, a particle moves from point A to point B under a constant force field along $\vec{AB}$,



the work done of $\vec{F}$ on this particle is given by

$$\vec{F}\cdot\vec{v}$$

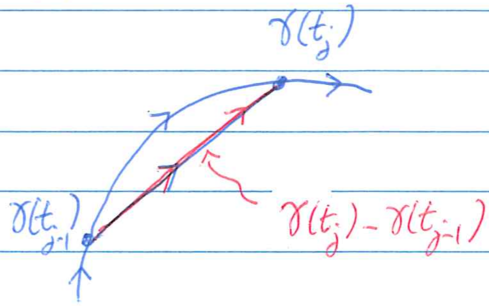
when $\vec{v}$ is the vector from A to B. Now, the work done of the particle along a curve C under a general force field can be defined via Riemann sums,



$$\int_C \vec{F}\cdot d\vec{r}$$

$$\sim \sum_j \vec{F}(\gamma(t_j^*))\cdot(\vec{\gamma}(t_j)-\vec{\gamma}(t_{j-1}))$$

$$= \sum_j \vec{F}(\gamma(t_j^*))\cdot \frac{\vec{\gamma}(t_j)-\vec{\gamma}(t_{j-1})}{|\vec{\gamma}(t_j)-\vec{\gamma}(t_{j-1})|}\Delta s_j$$



$$\Delta s_j = |\gamma(t_j)-\gamma(t_{j-1})|$$

$$\longrightarrow \int_C \vec{F}\cdot\vec{T}\, ds \quad \text{as } \|P\|\to 0.$$

Another formula of the line integral can be obtained by

$$\sum_j \vec{F}(\gamma(t_j^*))\cdot(\vec{\gamma}(t_j)-\vec{\gamma}(t_{j-1}))$$

$$= \sum_j \left[P(\gamma(t_j^*))(x(t_j)-x(t_{j-1})) + Q(\gamma(t_j^*))(y(t_j)-y(t_{j-1}))\right]$$

$$= \sum_j \left(P(\gamma(t_j^*))\, x'(t_j^{\star}) + Q(\gamma(t_j^*))\, y'(t_j^{\circ})\right)(t_j - t_{j-1})$$

$$\to \int_C \left(P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t)\right) dt$$

when $\vec{F}=(P,Q)$, $\gamma(t)=(x(t),y(t))$.

Therefore,

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t)\right) dt \qquad (n=2)$$

$$= \int_a^b \left(P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t) + R(\gamma(t))\, z'(t)\right) dt$$

$(n=3)$.

This is the most common way to evaluate the integral. It also motivates the new notations

$$\int_C P\,dx + Q\,dy\ , \quad \int_C P\,dx + Q\,dy + R\,dz .$$

e.g 4. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where

$C:\ \gamma : [0, \frac{\pi}{2}] \to (\cos t, \sin t, t)$

$\vec{F}(x,y,z) = (x, xy, z^3)$

$\gamma'(t) = (x'(t), y'(t), z'(t)) = (-\sin t, \cos t, 1)$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_C x\,dx + xy\,dy + z^3\,dz$$

$$= \int_0^{\frac{\pi}{2}} \cos t(-\sin t)\, dt + \int_0^{\frac{\pi}{2}} \cos t \sin t \cos t\, dt$$

$$+ \int_0^{\frac{\pi}{2}} t^3 \cdot 1\, dt$$

$$= \int_0^{\frac{\pi}{2}} -\cos t \sin t \, dt + \int_0^{\frac{\pi}{2}} \sin t \cos^2 t \, dt + \int_0^{\frac{\pi}{2}} t^3 \, dt$$

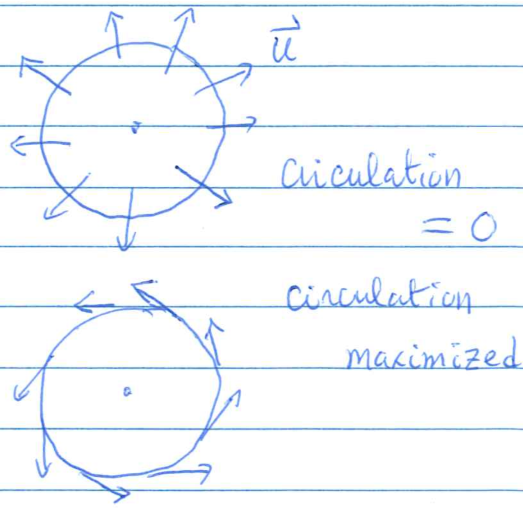
$$= -\frac{1}{2} + \frac{1}{3} + \frac{\pi^4}{64} = \frac{1}{6} + \frac{\pi^4}{64} . \#$$

Although the line integral of the second kind, that's, the line integral of a v.f. along an oriented curve, is motivate from work done, it has applications elsewhere.

Let C be a closed curve in $\mathbb{R}^2, \mathbb{R}^3$ and $\vec{u}$ a vector field on the region containing C. The line integral

$$\int_C \vec{u} \cdot d\vec{r}$$

is called the circulation of $\vec{u}$ around C. When $\vec{u}$ is the velocity of a fluid, $\vec{u} \cdot \vec{T}$ measures the amount of fluid moves along the curve C in unit time, hence the line integral measure the total amount of the fluid goes around the curve.



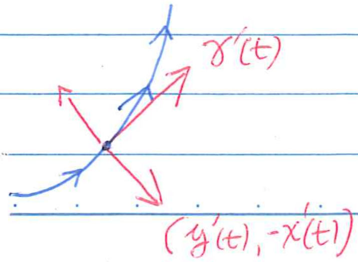
When $n = 2$, let γ parametrize C.

$$\gamma(t) = (x(t), y(t))$$

$$\gamma'(t) = (x'(t), y'(t))$$

The vectors $(-y'(t), x'(t))$, $(y'(t), -x'(t))$ satisfy

$$(-y'(t), x'(t)) \cdot \gamma'(t) = 0$$



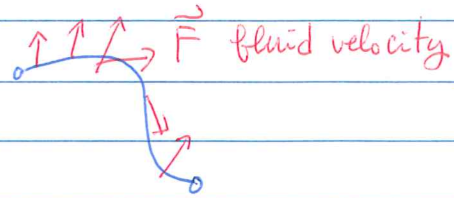
Use $\hat{n} = \dfrac{(-y'(t), x'(t))}{|\gamma'(t)|}$, or $\dfrac{(y'(t), -x'(t))}{|\gamma'(t)|}$

to denote the normal direction.

Let C be a curve in $\mathbb{R}^2$ (not nec. closed) with a choice of unit normal $\hat{n}$, the flux of the vector field $\vec{F}$ across C along $\hat{n}$ is defined to be

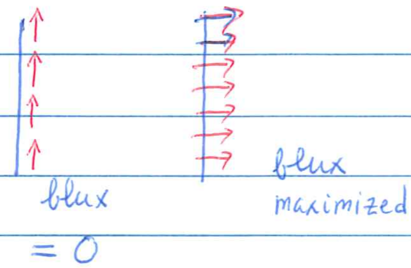
$$\int_C \vec{F}\cdot\hat{n}\, ds.$$

It measures the amount of fluid passing through C in unit time.



e.g. 5 Consider the v.f.

$$(ax+by,\ cx+dy),\quad a^2+b^2=1,\ c^2+d^2=1$$



on the circle (anti-clockwise)

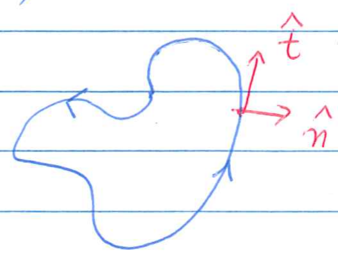
$$t\mapsto(\cos t,\sin t).$$

Find the circulation and flux of the v.f. along / across the circle.

$$\gamma(t)=(\cos t,\sin t),$$

$$\hat{t}=(-\sin t,\cos t)$$

$$\hat{n}=(\cos t,\sin t)$$ (for a closed curve, take $\hat{n}$ to be the outer one)



circulation:

$$\int_C \vec{F}\cdot\hat{t}\, ds=\int_0^{2\pi}(a\cos t+b\sin t,\ c\cos t+d\sin t)\cdot(-\sin t,\cos t)\,dt$$

$$= \int_0^{2\pi} \left(a\cos t(-\sin t) + b\sin t(-\sin t) + c\cos^2 t + d\sin t\cos t \right) dt$$

$$= (-b+c)\pi .$$

Therefore, the circulation is maximal when $a=0$, $b=-1$, $c=1$ and $d=0$. That is, the v.f. is $(-\sin t, \cos t)$. It is minimal at $b=1$ and $c=-1$ $(a=d=0)$, ie, v.f. is $(\sin t, -\cos t)$.

The flux

$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_0^{2\pi} (a\cos t + b\sin t, c\cos t + d\sin t)\cdot(\cos t, \sin t)\, dt$$

$$= (a+d)\pi .$$

So, the flux is maximal when v.f $(\cos t, \sin t)$ $(a=d=1, b=c=0)$ and minimal when v.f. $(-\cos t, -\sin t)$ $(a=d=-1, b=c=0)$.